NONCOMMUTATIVE PEAK INTERPOLATION REVISITED

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Abstract. We discuss noncommutative peak interpolation, which we have studied with coauthors in a long series of papers, and whose basic theory now appears to be approaching its culmination. This program developed from, and is based partly on, theorems of Hay and Read whose proofs were spectacular, but therefore inaccessible to an uncommitted reader. We give short proofs of these results, using recent progress in noncommutative peak interpolation, and conversely give examples of the use of these theorems in peak interpolation. For example, we prove a new noncommutative peak interpolation theorem that looks very useful.

1. Introduction

For us, an operator algebra is a norm closed algebra of operators on a Hilbert space. In 'noncommutative peak interpolation', one generalizes classical 'peak interpolation' to the setting of operator algebras, using Akemann's noncommutative topology [1, 2, 3]. In classical peak interpolation the setting is a subalgebra A of $C(K)$, the continuous scalar functions on a compact Hausdorff space K, and one tries to build functions in A which have prescribed values or behaviour on a fixed closed subset $E \subset K$ (or on several disjoint subsets). The sets E that 'work' for this are the p-sets, namely the closed sets whose characteristic functions are in $A^{\perp\perp}$. *Glicksberg's peak set theorem* characterizes these sets as the intersections of *peak sets*, i.e. sets of form $f^{-1}(\{1\})$ for $f \in A, \|f\| = 1$. The typical peak interpolation result, originating in results of E. Bishop, says that if $f \in C(K)$ is strictly positive, then the continuous functions on E which are restrictions of functions in A, and which are dominated by the 'control function' f on E, have extensions h in A satisfying $|h| \leq f$ on all of K (see e.g. II.12.5 in [13]). A special case of interest is when $f = 1$; for example when this is applied to the disk algebra one obtains the well known Rudin-Carleson theorem (see II.12.6 in [13]). It also yields 'Urysohn type lemmas' in which we find functions in A which are 1 on E and close to zero on a closed set disjoint from E. We discuss below generalizations of these results.

Noncommutative interpolation for C^*-algebras has been studied by many C^*-algebraists, and is a key application of Akemann's noncommutative topology. See particularly L.G. Brown's treatise [11]. For example Akemann's Urysohn lemma for C^*-algebras is a noncommutative interpolation result of a selfadjoint flavor, and this result plays a role in recent approaches to the important Cuntz semigroup [18]. Noncommutative peak interpolation for (possibly nonselfadjoint) operator algebras was introduced in the thesis of our student Damon Hay [15, 16]. Here we

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have a subalgebra A of a possibly noncommutative C^*-algebra B, and we wish to build operators in A which have prescribed behaviours with respect to Akemann's noncommutative generalizations of closed sets, which are certain projections q in B^{**}. In the case that $B = C(K)$, the characteristic function $q = \chi_E$ of an open or closed set E in K may be viewed as an element of $C(K)^{**}$ in a natural way since $C(K)^*$ is a certain space of measures on K. Via semicontinuity, it is natural to declare a projection $q \in B^{**}$ to be *open* if it is a increasing (weak*) limit of positive elements in B, and closed if its 'perp' $1 - q$ is *open* (see [1, 2]). Thus if $B = C(K)$ the open or closed projections are precisely the characteristic functions of open or closed sets.

Over the years we with coauthors (particularly Hay, Neal, and Read) have developed a number of noncommutative peak interpolation results, which when specialized to the case $B = C(K)$ collapse to classical peak interpolation theorems. Moreover, in the process of this investigation striking applications have emerged to the theory of one-sided ideals or hereditary subalgebras of operator algebras, the theory of approximate identities, noncommutative topology, noncommutative function theory, and other topics (see e.g. [5, 9, 10, 7, 8, 17, 20]). Current with the most recent version of [10] the peak interpolation program appears to be approaching its culmination. The basic theory seems now to be essentially complete, and we have a good idea of what works and what does not. What remains is further applications. Two of the most powerful results in the theory are Read's theorem on approximate identities [19], and Hay's main theorem in [16, 15]. These are foundational results in the subject, but the extreme depth of their proofs hindered their accessibility. In the present note we give short proofs of both of these results. The proofs are still quite nontrivial, but we have written them so as to be readable in full detail in an hour or less by a functional analyst. The crux of the proof is a special case of a new noncommutative peak interpolation theorem, the latter also proven here, generalizing the classical one mentioned above in the first paragraph of our paper (involving f and h). Below we will give examples of applications of both results to peak interpolation.

Turning to notation, the reader is referred for example to [6, 5, 9] for more details on some of the topics below if needed. We will use silently the fact from basic analysis that $X^{\perp\perp}$ is the weak* closure in Y^{**} of a subspace $X \subset Y$, and is isometric to X^{**}. Recall that X is an *M-ideal* in Y if $X^{\perp\perp} \oplus_\infty L = Y^{**}$ for a subspace L, and that in this case for any $y \in Y$ there exists $x \in X$ with the distance $d(y, X) = \|y - x\|$ (see [14]). For us a *projection* is always an orthogonal projection. An *approximately unital* operator algebra is one that has a contractive approximate identity (cai). If A is a nonunital operator algebra represented (completely) isometrically on a Hilbert space H then one may identify the unitization A^1 with $A + \mathbb{C}\, I_H$. If A is unital (i.e. has an identity of norm 1) we set $A^1 = A$. If A is an operator algebra then the second dual A^{**} is an operator algebra too with its (unique) Arens product, this is also the product inherited from the von Neumann algebra B^{**} if A is a subalgebra of a C^*-algebra B.

2. Read's theorem

In the following, A is an operator algebra with cai. Let C be any C^*-algebra generated by A, which has the same cai by [6, Lemma 2.1.7 (2)], and let $B = C^1$,

which is a C^*-algebra generated by A^1. Let e be the weak* limit of the cai in $(A^1)^{**}$, and let $q = 1 - e$. Both projections are in the center of B^{**}.

Lemma 2.1. *Suppose that A is an approximately unital operator algebra, and let q, e, C, B be as above. If $q \leq d$ for an invertible $d \in B$, then there exists an element $g \in A^1$ with $gq = qg = q$, and $g^* g \leq d$. Thus if A is nonunital and $c \in C_+$ with $\|c\| < 1$ then there exists an $a \in A$ with $|1 + a|^2 \leq 1 - c$.*

Proof. Let $f = d^{-\frac{1}{2}}$. Since the 'second perp' is the weak* closure, we have $(A^1 f)^{**} = (A^1 f)^{\perp\perp} = (A^1)^{\perp\perp} f$. Multiplication by the central projection $e = 1 - q$ (resp. by q) is a contractive projection on $(A^1)^{\perp\perp} f$ whose range is $e(A^1)^{\perp\perp} f = A^{\perp\perp} f = (Af)^{\perp\perp}$ (resp. is $qA^{\perp\perp} f$), which may be viewed as a subspace of $eB^{**} \oplus_\infty qB^{**}$. So Af is an M-ideal in $A^1 f$ as defined in the introduction, and, as we said there, there exists $y \in A$ such that $\|f - yf\| = d(f, Af)$. Since

$$A^1 f / Af \subset (A^1 f)^{**} / (Af)^{\perp\perp} = (A^1)^{\perp\perp} f / (e(A^1)^{\perp\perp} f) \cong q(A^1)^{\perp\perp} f,$$

we have $d(f, Af) = \|qf\| = \|fqf\|^{\frac{1}{2}} \leq 1$. Setting $g = 1 - y$ then $qg = gq = q$, and $\|gf\| \leq 1$, so that $fg^* gf \leq 1$ and so $g^* g \leq d$. $\square$

Remark. The last assertion of Lemma 2.1 is equivalent to the other assertion.

Theorem 2.2. (Read's theorem on approximate identities) *If A is an operator algebra with a cai, then A has a cai (e_t) with positive real parts, indeed satisfying $\|1 - 2e_t\| \leq 1$ for all t.*

Proof. Let q, e, C, B be as above. Then e is an open projection in the C^*-algebra sense with respect to B. Indeed, any increasing cai (b_t) for C is a net of positive elements in B increasing to e. Note that $q(1 - b_t) = (1 - b_t)q = q$. Consider the net $(f_s) = (\frac{1}{n} 1 + \frac{n-1}{n}(1 - b_t))$, which is a net of strictly positive elements f_s in $\mathrm{Ball}(B)$ with weak* limit q. Here $s = (t, n)$. We have $qf_s = f_s q = q$, and $f_s \geq q$. By Lemma 2.1, there exists $a_s \in A^1$ with $a_s q = qa_s = q$ and $a_s^* a_s \leq f_s \leq 1$. We have $(a_s - q)^*(a_s - q) = a_s^* a_s - q \leq f_s - q \to 0$ weak*. Using the universal representation we may view B^{**} as a von Neumann algebra in $B(H)$ in such a way that the weak* topology of B^{**} coincides with the σ-weak topology. Then $f_s \to q$ WOT, and so for $\zeta, \eta \in \mathrm{Ball}(H)$ we have

$$|\langle (a_s - q)\zeta, \eta \rangle|^2 \leq \|(a_s - q)\zeta\|^2 = \langle (a_s - q)^*(a_s - q)\zeta, \zeta \rangle \leq \langle (f_s - q)\zeta, \zeta \rangle \to 0.$$

So $a_s \to q$ WOT, and hence $a_s \to q$ weak*. If $u_s = 1 - a_s$ then $eu_s = u_s e = u_s$, so $u_s \in A$, indeed u_s is in the convex subset $\mathfrak{F}_A = \{a \in A : \|1 - a\| \leq 1\}$ of $2\mathrm{Ball}(A)$. Also $u_s \to e$ weak*, so that (xu_s) and $(u_s x)$ converge weakly to x for any $x \in A$. We now apply a standard convexity argument: for $x_1, \cdots, x_m \in A$, the norm and weak closures of the convex set

$$F = \{(x_1 u - x_1, \cdots, x_m u - x_m, ux_1 - x_1, \cdots, ux_m - x_m) : u \in \mathfrak{F}_A\}$$

coincide by Mazur's theorem, and contain 0, from which it follows that A has a bounded approximate identity (e_r) in $\mathfrak{F}_A$ (see e.g. the last part of the first paragraph of the proof of [5, Theorem 6.1] for more details if needed).

A result of the author and Read states that $\frac{1}{2}\mathfrak{F}_A$ is closed under nth roots [9, Proposition 2.3], so $((\frac{1}{2}e_r)^{\frac{1}{n}}) \subset \frac{1}{2}\mathfrak{F}_A$. Since $(\frac{1}{2}e_r)^{\frac{1}{n}} e_r \to e_r$ with n (see below), it is easy to see that $((\frac{1}{2}e_r)^{\frac{1}{n}})$ is a cai for A from $\frac{1}{2}\mathfrak{F}_A$. For example, if $a \in A$, then

$$\|a - (\frac{1}{2}e_r)^{\frac{1}{n}} a\| \leq \|a - e_r a\| + \|(e_r - (\frac{1}{2}e_r)^{\frac{1}{n}} e_r)a\| + \|(\frac{1}{2}e_r)^{\frac{1}{n}}(a - e_r a)\| \to 0$$

with n and r, since the middle term here is bounded by $2\|a\|$ times

$$\frac{1}{2}\|e_r - (\frac{1}{2}e_r)^{\frac{1}{n}}e_r\| \leq \|p_n\|_{\bar{\mathbb{D}}} \to 0$$

with n, using von Neumann's inequality by applying to $1 - e_t$ the function $p_n(z) = 1 - z - (1-z)^{1+\frac{1}{n}}$ on the unit disk $\mathbb{D}$. $\square$

A sample application of Read's theorem to noncommutative peak interpolation: the point in the last proof where we show that A has a bounded approximate identity in $\mathfrak{F}_A$ already solves the main open question that arose in Hay's thesis [15, 16], namely the validity of the noncommutative version of Glicksberg's peak set theorem mentioned in the first paragraph of the paper, by [5, Theorem 6.1] and the surrounding discussion. That is, the closed projections in B^{**} which lie in $A^{\perp\perp}$, are precisely the 'infs' of *peak projections*. The latter are Hay's noncommutative generalization of peak sets and have many characterizations in the papers cited below. The following evocative characterization is proved at the end of the introduction of [10], the converse holding by [8, Lemma 3.1]: For any operator algebra A, the peak projections are the weak* limits of a^n for $a \in \mathrm{Ball}(A)$ in the cases that such limit exists.

3. Peak interpolation and Hay's theorem

The following general functional analytic lemma is due to the author and Hay [16, Proposition 3.1] (its proof follows the lines of [13, Lemma II.12.3]).

Lemma 3.1. *Let X be a closed subspace of a unital C^*-algebra B, and let $q \in B^{**}$ be a closed projection such that $(qx)(\varphi) = 0$ whenever $\varphi \in X^\perp, x \in X$. Let $I = \{x \in X : qx = 0\}$. Then the distance $d(x, I) = \|qx\|$ for all $x \in X$.*

The following 'approximate interpolation' result ([16, Proposition 3.2]) follows easily from Lemma 3.1 following the lines of [13, Lemma II.12.4].

Corollary 3.2. *If q satisfies the conditions in the last result, and keeping the notation there, if d is a positive invertible element of B with $d \geq a^*qa$ for some $a \in A$, and if $\epsilon > 0$, then there exists $b \in X$ with $qb = qa$ and $b^*b \leq d + \epsilon 1$.*

Theorem 3.3. (Hay's theorem on one-sided ideals) *If A is a unital subalgebra of a unital C^*-algebra B, then the right ideals J in A which have a left contractive approximate identity, are precisely the right ideals $\{a \in A : a = pa\}$ for an open projection $p \in B^{**}$ which lies in $A^{\perp\perp}$. If these hold then $J^{\perp\perp} = pA^{**}$.*

Proof. As explained in [16], this is easy and standard functional analysis, most of it working in any Arens regular Banach algebra, except for the following Claim: if an open projection $p \in B^{**}$ lies in $A^{\perp\perp}$, then p is a weak* limit of a net $(x_s) \subset A$ satisfying $px_s = x_s$. For example, if the Claim holds then p is in the weak* closure of the right ideal $I = \{a \in A : a = pa\}$, and is a left identity for that weak* closure, and then it is well known (see e.g. [6, Proposition 2.5.8]) that I has a left cai.

To prove the Claim, note that $q = 1-p$ is closed. As in the first lines of the proof of Theorem 2.2, there is a net in B of strictly positive $f_t \searrow q$. Let $f^s = f_t + \frac{1}{n}1$, where $s = (t, n)$. By Corollary 3.2 with $a = 1$, there exists $a_s \in A$ with $qa_s = q$ and $a_s^*a_s \leq f^s$. Then $a_s \to q$ weak* as in the proof of Theorem 2.2, so $x_s = 1 - a_s$ satisfies $px_s = x_s \to p$ weak*. $\square$

Lemma 2.1 happens to be a special case of the following peak interpolation result, which uses Hay's theorem:

Theorem 3.4. *Suppose that A is an operator algebra (not necessarily approximately unital), a subalgebra of a unital C^*-algebra B. Identify $A^1 = A + \mathbb{C}\,1_B$. Suppose that q is a closed projection in $(A^1)^{**}$. If $b \in A$ with $bq = qb$, and $qb^*bq \leq qd$ for an invertible positive $d \in B$ which commutes with q, then there exists an element $g \in A$ with $gq = qg = bq$, and $g^*g \leq d$.*

Proof. Let $\tilde{D} = (1-q)(A^1)^{**}(1-q) \cap A^1$, let C be the closed subalgebra of A^1 generated by $\tilde{D}, b$, and 1, and let $D = \{x \in C \cap A : qx = 0\} \subset A$. By Hay's theorem above $1-q$ is a limit of $x_s = (1-q)x_s \in A^1$, and by symmetry $1-q$ is a limit of $z_t = z_t(1-q) \in A^1$. Hence $1-q$ is a limit of $x_s z_t \in \tilde{D}$, that is $1-q \in \tilde{D}^{\perp\perp} \subset C^{\perp\perp}$. Thus $q \in C^{\perp\perp}$. Clearly $\tilde{D}^{\perp\perp} \subset (1-q)C^{\perp\perp}$. Conversely, since $\tilde{D}$ is an ideal in C, so that $\tilde{D}^{\perp\perp}$ is an ideal in $C^{\perp\perp}$, we have $(1-q)C^{\perp\perp} \subset \tilde{D}^{\perp\perp}$. So

$$(\tilde{D}f)^{\perp\perp} = \tilde{D}^{\perp\perp}f = (1-q)C^{\perp\perp}f = (1-q)(Cf)^{\perp\perp},$$

and so $\tilde{D}f$ is an M-ideal in Cf, using the fact that q is a central projection in $C^{\perp\perp}$. The associated L-projection P onto the subspace $(\tilde{D}f)^{\perp}$ of $(Cf)^*$, is multiplication by q. Let $x \in (C \cap A)f$ and $\varphi \in ((C \cap A)f)^{\perp}$, and let (c_t) be a net in C with weak* limit q. Then $q\varphi(x) = \lim_t \varphi(c_t x) = 0$, since $c_t x \in (C \cap A)f$ (because $C(C \cap A) \subset (C \cap A)$). We will make two deductions from this. First, $P(((C \cap A)f)^{\perp}) \subset ((C \cap A)f)^{\perp}$. So by [14, Proposition I.1.16], we have that Df is an M-ideal in $(C \cap A)f$. Second, we deduce from Lemma 3.1 with $X = (C \cap A)f$, that $d(x, Df) = \|qx\|$ for all $x \in (C \cap A)f$. So $d(bf, Df) = \|qbf\| = \|fqb^*bqf\|^{\frac{1}{2}} \leq 1$. By the distance formula in the introduction there exists $yf \in Df$ such that $\|bf - yf\| \leq 1$. Setting $g = b - y$ then $qg = gq = qb$, and $\|gf\| \leq 1$, so that $fg^*gf \leq 1$ and $g^*g \leq d$. □

Let us see that Theorem 3.4 implies the classical peak interpolation result in the first paragraph of the paper (and therefore a 'commutativising' of the proof above gives a new and quick proof of that classical result). If $B = C(K)$, and if E is a peak or p-set in K for $A \subset C(K)$, then by what we said in in the first two paragraphs of the introduction, the characteristic function of E may be viewed as a closed projection q in $A^{**} = A^{\perp\perp} \subset C(K)^{**}$. The condition that $qb^*bq \leq qd$ is saying precisely that the strictly positive function $d \in C(K)$ dominates $|b|^2$ on E. Thus Theorem 3.4 gives $g \in A$ with $|g|^2 \leq d$ on K, and $g = b$ on E (since $g\chi_E = b\chi_E$).

In [10] it is shown that one cannot drop the condition $bq = qb$ in the last result. It is easy to see one also cannot drop the condition $dq = qd$ (a counterexample: $A = \mathbb{C}\,q, b = q$ for a nontrivial projection $q \in M_2$).

The noncommutative peak interpolation theorem 3.4 should have many applications. For example in the last section of [10] a special case of it is used to develop the theory of compact projections in algebras not necessarily having any kind of approximate identity. Using this special case we obtain a generalization of Glicksberg's peak set theorem mentioned in the first paragraph of the paper. That is, even if A has no approximate identity then the compact projections relative to A, that is the closed projections with respect to a unitization, which lie in $A^{\perp\perp}$, are precisely the 'infs' of the peak projections discussed at the end of the last section. If A is separable then every compact projection relative to A is a peak projection. We also obtain noncommutative Urysohn type lemmas in that setting (that is, given

compact q relative to A, with $q \leq p$ open, there exists $a \in A$ with $aq = qa = q$ and a 'small' on $1 - p$). Indeed we show that these results follow from the case $d = 1$ of Theorem 3.4 and ideas in the proof of the case of these results when A is approximately unital from [8, 9] (which in turn uses Read's theorem).

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